Extension of Dirac Theorem on Hamiltonicity *

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In this study, we focus on the classical Dirac theorem, which asserts that every graph on \( n \) vertices with minimum degree at least \( \lceil n/2 \rceil \) is Hamiltonian. This lower bound of \( \lceil n/2 \rceil \) on minimum degree of a graph strictly holds. In this paper, we extend the classical Dirac theorem by classifying the only graphs without a Hamiltonian cycle when the minimum degree is at least \( \lfloor n/2 \rfloor \). Our proof is constructive and hence may lead to a polynomial time algorithm, which is left as a future work.

1 Introduction

A Hamiltonian cycle of a graph is a cycle which passes through every vertex of the graph exactly once, and a graph is Hamiltonian if it contains a Hamiltonian cycle. Finding a Hamiltonian cycle in a graph is one of the important problems in graph theory, and has been studied for years. Karp [4] proved that the problem of determining whether a Hamiltonian cycle exists in a given graph is NP-complete. However, in the past years some important sufficient conditions have been found. For instance, in [6] Ore proved that for all distinct nonadjacent pairs of vertices \( u, v \) of a graph \( G \) if the sum of degrees of \( u \) and \( v \) is at least the order of \( G \), then \( G \) is Hamiltonian. One vital sufficient condition proved by Dirac [2] is that every graph on \( n \) vertices with minimum degree at least \( \lfloor n/2 \rfloor \) is Hamiltonian. This lower bound on the minimum degree of a graph strictly holds, so no smaller minimum degree can be sufficient for hamiltonicity of a given graph in general. But we show that except two families of graphs this minimum degree bound can be lowered to \( \lfloor n/2 \rfloor \).

Furthermore, some additional sufficient conditions have been found for special graph classes. Nash-Williams [5] proved that every \( k \)-regular graph on \( 2k + 1 \) vertices is Hamiltonian. In [5], he also proved that a 2-connected graph of order \( n \) with minimum degree at least \( \max\{(n + 2)/3, \beta\} \), where \( \beta \) is independence number, is Hamiltonian. Although this last result is more general than others, since finding the independence number of a graph is in general NP-hard, this result does not give an efficient algorithm to determine the hamiltonicity of any given graph. In contrast, here we give a constructive characterization without using independence.

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number. We extend the classical Dirac theorem on Hamiltonian cycle to the case where the minimum degree is at least \(\lfloor n/2 \rfloor\) by classifying non-Hamiltonian graphs under this condition.

We adopt [7] and [3] for terminology and notation not defined here. A graph \(G = (V, E)\) is given by a pair of a vertex set \(V = V(G)\) and a edge set \(E = E(G)\). In this work, we consider only simple graphs, which have no loops or multiple edges. Particularly, \(G_n\) denotes not necessarily connected simple graph on \(n\) vertices. We use \(|V(G)|\) to denote the order of \(G\) and \(N(v)\) to denote the neighborhood of a vertex \(v\) of \(G\). In addition, \(d(v)\) denotes degree of a vertex \(v\) of \(G\) and \(\delta(G)\) denotes the minimum degree of \(G\). The distance \(d(u, v)\) between two vertices \(u\) and \(v\) is the length of a shortest path joining \(u\) and \(v\); and the diameter of \(G\), denoted by \(d(G)\), is the maximum distance among all pairs of vertices of \(G\). Given a graph \(G\) with \(n\) vertices, the closure \(cl(G)\) of \(G\) is uniquely constructed from \(G\) by repeatedly adding a new edge \(uv\) connecting a nonadjacent pair of vertices \(u\) and \(v\) with \(d(v) + d(u) \geq n\) until no more pairs with this property can be found.

Moreover, there are binary operations which create a new graph from two initial graphs \(G(V, E)\) and \(G'(V', E')\). The union of two graphs \(G(V, E)\) and \(G'(V', E')\) is the union of their vertex and edge sets, denoted by \(G \cup G' = (V \cup V', E \cup E')\). When \(V\) and \(V'\) are disjoint, their union is referred to as the disjoint union. The join of graphs \(G\) and \(G'\) is the disjoint union graph \(G \cup G'\) together with all the edges joining \(V\) and \(V'\), denoted by \(G + G'\). The complete graphs can be seen as the join of two complete graphs, \(K_{n_1+n_2} = K_{n_1} + K_{n_2}\); and the complete bipartite graphs can be seen as the join of empty graphs, \(K_{n,m} = K_n + K_m\).

The classical Dirac theorem, which we will extend as the main result of this work, is stated as follows:

**Theorem 1.** (Dirac [2]) If \(G\) is a graph of order \(n \geq 3\) such that \(\delta(G) \geq \lfloor n/2 \rfloor\), then \(G\) is Hamiltonian.

We now present the main theorem of this paper:

**Theorem 2.** (Extension of Dirac Theorem) Let \(G\) be a graph of order \(n \geq 3\) such that \(\delta(G) \geq \lfloor n/2 \rfloor\). Then \(G\) is Hamiltonian unless \(G\) is the graph \(K_{\lfloor n/2 \rfloor} \cup K_{\lfloor n/2 \rfloor}\) with one common vertex or the graph \(\bar{K}_{\lfloor n/2 \rfloor} + G_{\lfloor n/2 \rfloor}\) for odd \(n\).

## 2 Proof of the Main Theorem

In this section we constructively prove the main theorem. In contrast to Theorem 3, we extend the classical Dirac theorem to the case \(\delta(G) \geq \lfloor n/2 \rfloor\) by classifying non-Hamiltonian graphs without using the independence number under this condition.

The following results will be used to establish the proof of the main theorem:

**Lemma 3.** (Nash-Williams [5]) Let \(G\) be a 2-connected graph of order \(n\) with \(\delta(G) \geq \max\{(n + 2)/3, \beta\}\), where \(\beta\) is the independence number of \(G\). Then \(G\) is Hamiltonian.

**Lemma 4.** (Bondy-Chvátal [1]) A graph \(G\) is Hamiltonian if and only if its closure \(cl(G)\) is Hamiltonian.

We prove the main result of this work as follows:

**Proof of Theorem 2.** For \(n = 2r\) where \(r \in \mathbb{Z}^+\), the result holds by Theorem 1. Hence, we assume that \(n = 2r + 1\) and \(\delta(G) \geq r\). First, we add a vertex \(y\) to the graph \(G\) and connect it
to all other vertices. This new graph $G'$ has $|V(G')| = 2r + 2$ and $\delta(G') \geq r + 1$. By Theorem 1, it has a Hamiltonian cycle. After removing $y$, we still have a Hamiltonian path $P$ in $G$, say $P = (x_0, ..., x_{2r})$.

Suppose $G$ has no Hamiltonian cycle. That is, $x_0$ and $x_{2r}$ are not adjacent. W.l.o.g, if $d(x_0) > r$ or $d(x_{2r}) > r$, then $x_0$ and $x_{2r}$ are adjacent in closure of $G$. Hence, the closure $cl(G)$ is Hamiltonian and therefore $G$ is Hamiltonian by Theorem 4. That is, we can assume that $d(x_0) = d(x_{2r}) = r$.

Now, we observe the following facts:

1. If $x_0$ is adjacent to $x_i$, then $x_{2r}$ is not adjacent to $x_{i-1}$. Otherwise, the closed trail $x_0x_1...x_{i-1}x_2x_{2r-1}...x_{2r-2}...x_1x_0$ yields a Hamiltonian cycle.

2. If $x_0$ is not adjacent to $x_i$, then $x_{2r}$ is adjacent to $x_{i-1}$. By the first fact, $x_{2r}$ can be adjacent to vertices whose successive vertices in $P$ are not adjacent to $x_0$. Since the number of vertices whose successive vertices in $P$ are not adjacent to $x_0$ is $r$ and $d(x_{2r}) = r$, $x_{2r}$ has to be adjacent to the all such vertices.

3. Every pair of non-adjacent vertices $x_i$ and $x_j$ have at least one common neighbor where $0 \leq i, j \leq 2r$. Notice that the diameter $d(G) = 2$ since $|V(G)| = 2r + 1$ and $d(x_i), d(x_j) \geq r$ for any $0 \leq i, j \leq 2r$.

Then, the following two cases arise:

**Case 1:** $N(x_0) \cup N(x_{2r}) = V(G)$ By assumption and the third fact, $x_0$ and $x_{2r}$ have exactly one common neighbor $x_k$. Then $x_{k-1}$ is not adjacent to $x_{2r}$ but adjacent to $x_0$. Proceeding in the same way, we conclude that $x_0$ is adjacent to all vertices $x_k$ through $x_k$ and $x_{2r}$ is adjacent to all vertices $x_k$ through $x_{2r-1}$. Since both vertices $x_0$ and $x_{2r}$ have degree $r$, we conclude that $k = r$. Hence, there is an $i_0$ with $1 < i_0 \leq r$ such that $x_{i_0}$ is adjacent to $x_i$ for all $r + 1 \leq i < 2r$. If there is a $x_{i_0} \neq x_r$ for any $r + 1 \leq i < 2r$, then the cycle $x_{i_0}x_{i_0-1}...x_0x_{i_0}x_{i_0+1}x_{i_0+2}...x_{i-1}x_2x_{2r-1}...x_1x_0$ is a Hamiltonian cycle in $G$. If there is no $x_{i_0} \neq x_r$ for all $r + 1 \leq i < 2r$, then we have a non-Hamiltonian graph $K_{[n/2]} \cup K_{[n/2]}$ with common vertex $x_r$.

**Case 2:** $N(x_0) \cup N(x_{2r}) \neq V(G)$ Then, there is an $i_0$ with $1 < i_0 < 2r - 1$ such that $x_{i_0+1}$ is adjacent to $x_0$, but $x_{i_0}$ is not. By the second fact, $x_{i_0-1}$ must be adjacent to $x_{2r}$. Hence, we have a $(2r)$-cycle $x_{i_0-1}x_{i_0-2}...x_0x_{i_0+1}x_{i_0+2}...x_{2r}x_{i_0-1}$ not containing $x_{i_0}$. W.l.o.g, let $C = (y_1, y_2, ..., y_{2r})$ be the $(2r)$-cycle and $y_0$ be the remaining vertex. Note that $C$ is a maximum cycle in $G$ due to the assumption that $G$ has no Hamiltonian cycle. It implies that $y_0$ cannot be adjacent to two consecutive vertices on $C$. Otherwise, $C$ is not a maximum cycle and there exists a Hamiltonian cycle. Therefore, $d(y_0) = r$ and $y_0$ must be adjacent to every second vertex on $C$. W.l.o.g, let the second vertices on $C$ be $y_1, y_3, ..., y_{2r-1}$. That is, $y_0$ is adjacent to all vertices with odd index and non-adjacent to any vertex with even index. Observe that replacing $y_{2i}$ by $y_0$ gives another maximum cycle $C'$ where $1 \leq i \leq r$, and then $d(y_{2i}) = r$ by the above argument on $y_0$. Therefore, every vertex with even index is adjacent to every vertex with odd index, and non-adjacent to any vertex with even index. Hence, we get the graph $\hat{K}_{[n/2]} + G_{[n/2]}$ where the vertices with even index form the empty graph $\hat{K}_{[n/2]}$ and the vertices with odd index form a not necessarily connected graph $G_{[n/2]}$. Notice that the graph $\hat{K}_{[n/2]} + G_{[n/2]}$ is a non-Hamiltonian graph since the order of $\hat{K}_{[n/2]}$ is larger than the order of $G_{[n/2]}$. 

□
3 Conclusion

In this paper, we have extended the classical Dirac theorem to the case where the minimum degree is at least $\lfloor n/2 \rfloor$ by classifying non-Hamiltonian graphs under this condition. Our proof is constructive and hence, as a future work, we plan to design a polynomial time algorithm which produces a Hamiltonian cycle in a given graph satisfying our condition, if exists, or says it is non-Hamiltonian.

References


